

Mixed Form of Ambiguous and Unambiguous Discriminations

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Abstract

In this paper, we introduce a mixed form of ambiguous and unambiguous quantum state discriminations, and show that the mixed form has higher success probability than the unambiguous quantum state discriminations.

Key words. Quantum state; quantum measurement; quantum state discrimination.

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1 Quantum state discrimination

Let \mathcal{H} be a finite dimensional complex Hilbert space. A quantum state ρ of some quantum system, described by \mathcal{H} , is a positive semi-definite operator of trace one, in particular, for each unit vector $|\psi\rangle \in \mathcal{H}$, the operator $\rho = |\psi\rangle\langle\psi|$ is said to be a *pure state*. We can identify the pure state $|\psi\rangle\langle\psi|$ with the unit vector $|\psi\rangle$. The set of all quantum states on \mathcal{H} is denoted by $D(\mathcal{H})$.

A quantum measurement on the quantum system \mathcal{H} is a family of operators $\{M_x\}_{x \in \Gamma}$ which are indexed by some classical labels x corresponding to the classical outcomes of the measurement. These operators satisfy ([1, 2, 3]):

$$\forall x : M_x \geq 0, \quad \sum_x M_x = \mathbb{1},$$

together with $\{A_x\}$ such that $M_x = A_x^\dagger A_x$. Given a quantum state ρ and a quantum measurement $\{M_x\}$, a probability distributive $p = (p_x)$ and a conditional state $\rho_{A|x}$ given outcome x are induced as following:

$$\rho_{A|x} = p_x^{-1} A_x \rho A_x^\dagger, \quad p_x = \text{Tr}(M_x \rho).$$

The carriers of information in quantum communication and quantum computing are quantum systems, the information is encoded in a set of states on those systems. After processing the information, Alice transmitting it to receiver Bob. Bob has to determine the output state of the system by performing quantum measurements. If given states $\{\rho_i\}_{i \in \Sigma}$ with orthogonal supports, then it is easy to devise a quantum measurement that discriminates them without any error. However, if the states $\{\rho_i\}_{i \in \Sigma}$ are non-orthogonal, then a perfect discrimination is impossible. It is important to find the best quantum measurement to distinguish the non-orthogonal states with the smallest possible error.

Now, ones have two way for discriminating non-orthogonal states, if the number $|\Gamma|$ of possible outcomes for quantum measurement $\{M_x\}_{x \in \Gamma}$ is equal to the number $|\Sigma|$ of states in the discriminating states, then it is called the ambiguous quantum measurement. If $|\Gamma| = |\Sigma| + 1$ and ones can identify perfectly each state ρ_i for $|\Sigma|$ measurement outcomes, but, there is a measurement outcome leads to an inconclusive result ([4]), then it is called the unambiguous quantum measurement.

Henceforth, for ambiguous quantum measurement, we identify the measurement outcome with the corresponding state, thus, the outcomes set Γ is Σ , for unambiguous quantum measurement, we identify the measurement outcome with the corresponding state, thus, the outcome set Γ is $\Sigma \cup \{0\}$, that is, for unambiguous quantum measurement, if the outcome is $i \in \Sigma$, then Bob is certain that the state is ρ_i , whereas if the outcome is 0, then he cannot decide what it is. Therefore, if $\{M_i\}_{i \in \Sigma \cup \{0\}}$ is an unambiguous quantum measurement, then for any $i, j \in \Sigma$, $\text{Tr}(M_i \rho_i) > 0$ and when $i \neq j$, $\text{Tr}(M_j \rho_i) = 0$.

Let us consider an ensemble $\{\rho_i, p_i\}_{i \in \Sigma}$ of states $\{\rho_i\}_{i \in \Sigma}$ with prior probability distribution $p = (p_i)$. Then for each ambiguous quantum measurement $M = \{M_i\}_{i \in \Sigma}$, the success probability of all quantum states $\{\rho_i\}_{i \in \Sigma}$ can be discriminated is ([4])

$$P_{suc}^{amb} = \sum_{i \in \Sigma} p_i \text{Tr}(M_i \rho_i).$$

For each unambiguous quantum measurement $M = \{M_i\}_{i \in \Sigma \cup \{0\}}$, the success probability of all quantum states $\{\rho_i\}_{i \in \Sigma}$ can be discriminated is

$$P_{suc}^{una} = \sum_{i \in \Sigma} p_i \text{Tr}(M_i \rho_i) = 1 - \sum_{i \in \Sigma} p_i \text{Tr}(M_0 \rho_i).$$

If the probability $p_0 = \sum_i p_i \text{Tr}(M_0 \rho_i)$ of occurrence of the inconclusive outcome is minimized, then the quantum measurement is said to be an optimal measurement.

Example 1.1. (RRA scheme, [5]) Let $\mathcal{H}_1 = \mathbb{C}^2$, $\{|0\rangle, |1\rangle\}$ be its orthogonal basis, $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. Consider two non-orthogonal quantum states $|\psi_+\rangle, |\psi_-\rangle \in \mathcal{H}_1$ are randomly prepared with a priori probability distributive $p = (p_+, p_-)$. In order to discriminate the two states $|\psi_+\rangle, |\psi_-\rangle$, taking an auxiliary qubit system \mathcal{H}_A , two complex numbers c_+, c_- with $\overline{c_+}c_- = \langle\psi_-|\psi_+\rangle$, and prepare a quantum state $|k_a\rangle$ in \mathcal{H}_A , $\{|0_a\rangle, |1_a\rangle\}$ is an orthonormal basis of \mathcal{H}_A , then couple \mathcal{H}_1 to \mathcal{H}_A by a joint unitary transformation U_1 :

$$\begin{aligned} U_1|\psi_+\rangle|k_a\rangle &= \sqrt{1-|c_+|^2}|+\rangle|0_a\rangle + c_+|0\rangle|1_a\rangle, \\ U_1|\psi_-\rangle|k_a\rangle &= \sqrt{1-|c_-|^2}|-\rangle|0_a\rangle + c_-|0\rangle|1_a\rangle. \end{aligned} \quad (1.1)$$

After the joint transformation, the quantum state we consider in discriminating is given by

$$\rho_1 = \sum_{i=+,-} p_i U_1(|\psi_i\rangle\langle\psi_i| \otimes |k_a\rangle\langle k_a|) U_1^\dagger. \quad (1.2)$$

Note that if we perform a von Neumann measurement $\{|0_a\rangle\langle 0_a|, |1_a\rangle\langle 1_a|\}$ on the auxiliary system, then the quantum state ρ_1 will collapse to either $|0_a\rangle\langle 0_a|$ or $|1_a\rangle\langle 1_a|$. If the system collapses to $|0_a\rangle\langle 0_a|$, we will discriminate successfully the original state since we can distinguish deterministically the two orthogonal states $|\pm\rangle$ in (1.1). However, we fail if the system collapses to $|1_a\rangle\langle 1_a|$. Thus, we can design a unambiguous quantum measurement $\Pi_1 = \{\pi_i\}_{i=+,-,0}$ on the quantum system $\mathcal{H}_1 \otimes \mathcal{H}_A$ as follows:

$$\pi_+ = |+\rangle\langle +| \otimes |0_a\rangle\langle 0_a|, \quad \pi_- = |-\rangle\langle -| \otimes |0_a\rangle\langle 0_a| \quad \text{and} \quad \pi_0 = \mathbb{1}_{\mathcal{H}_1} \otimes |1_a\rangle\langle 1_a|,$$

it will unambiguous discriminate the quantum states $|\psi_+\rangle|k_a\rangle$ and $|\psi_-\rangle|k_a\rangle$, therefore $|\psi_+\rangle$ and $|\psi_-\rangle$ are unambiguous discriminated, too.

The RRA scheme is extended to the case with three non-orthogonal states in \mathbb{C}^3 , that is:

Example 1.2. ([6]) Let $\mathcal{H}_2 = \mathbb{C}^3$, $\{|0\rangle, |1\rangle, |2\rangle\}$ be its orthogonal basis. Ones randomly prepared three nonorthogonal states $\{|u_i\rangle : i = 0, 1, 2\}$ with a priori probability distributive $p = (p_i)$, and these states satisfy that $\langle u_i | u_{j \neq i} \rangle = \gamma_{ij}$. In order to discriminate the three states $\{|u_i\rangle : i = 0, 1, 2\}$, we prepare $\{|\phi_i\rangle : i = 0, 1, 2\} \subseteq \mathcal{H}_2$, and taking complex numbers α_i, α_j such that $\overline{\alpha_i}\alpha_j\langle\phi_i|\phi_j\rangle = \gamma_{ij}$, then we couple the original system \mathcal{H}_2 to \mathcal{H}_A by the following joint unitary transformation U_2 :

$$U_2|u_i\rangle|k_a\rangle = \sqrt{1-|\alpha_i|^2}|i\rangle|0_a\rangle + \alpha_i|\phi_i\rangle|1_a\rangle, \quad (1.3)$$

where $i = 0, 1, 2$.

If we perform the von Neumann measurement

$$\begin{aligned}\pi_0 &= |0\rangle\langle 0| \otimes |0_a\rangle\langle 0_a|, & \pi_1 &= |1\rangle\langle 1| \otimes |0_a\rangle\langle 0_a|, & \pi_2 &= |2\rangle\langle 2| \otimes |0_a\rangle\langle 0_a| \\ \text{and} & & \pi_0 &= \mathbb{1}_{\mathcal{H}_2} \otimes |1_a\rangle\langle 1_a|\end{aligned}$$

on the quantum system $\mathcal{H}_2 \otimes \mathcal{H}_A$, then those three states $\{|u_i\rangle\}_{i=0,1,2}$ can be unambiguous discriminated.

Now, we assume $p_2 \geq p_1 \geq p_0$, and let

$$\begin{aligned}\gamma_1 &= \sqrt{p_1}/(\sqrt{p_2} - \sqrt{p_1}), \\ \gamma_2 &= \sqrt{p_0}/(\sqrt{p_2} - \sqrt{p_1}).\end{aligned}$$

In ([6]), the authors showed that if $\langle \psi_i | \psi_{j \neq i} \rangle = \gamma_{ij} = \gamma$, then the maximal success probabilities of unambiguous discrimination are:

- (1). If $\gamma_2 \geq 1$, then $P_{suc,max}^{una} = 1 - \gamma$,
- (2). If $\gamma \geq \gamma_1$, then $P_{suc,max}^{una} = 1 - p_0 - p_1 - 2p_2\gamma^2/(\gamma + 1)$,
- (3). If $\gamma_1 \geq \gamma \geq \gamma_2$, then $P_{suc,max}^{una} = 1 - p_0 - 2\sqrt{p_1 p_2}\gamma - (\sqrt{p_2} - \sqrt{p_1})^2\gamma^2$,
- (4). If $1 \geq \gamma_2 \geq \gamma$, then $P_{suc,max}^{una} = 1 - 2(\sqrt{p_1 p_2} + \sqrt{p_0 p_2} - \sqrt{p_0 p_1})\gamma$.

In this paper, for three quantum states discrimination, we introduce a mixed form of ambiguous and unambiguous quantum state discriminations, and show that the mixed form has higher success probability than the unambiguous quantum state discriminations.

2 Mixed form of ambiguous and unambiguous discriminations

Firstly, we consider a special case, that is, let $\mathcal{H}_2 = \mathbb{C}^3$ and prepare three states $\{|u_i\rangle\}_{i=0,1,2}$ in \mathcal{H}_2 with a priori probability distribution $p = (p_i)$. We assume that $\langle u_2 | u_0 \rangle = \langle u_2 | u_1 \rangle = \gamma \neq 0$, $\langle u_0 | u_1 \rangle = 0$, where γ is a real number. In order to discriminate the three states $\{|u_i\rangle\}$, we define

$$|v_i\rangle \equiv |u_i\rangle |k_a\rangle, i = 0, 1, 2.$$

Taking two states $|\psi_0\rangle, |\psi_1\rangle$ satisfying $\langle v_2 | \psi_0 \rangle = \langle v_2 | \psi_1 \rangle = 0$ and

$$\begin{aligned}|v_0\rangle &= \sqrt{1 - \gamma^2}|\psi_0\rangle + \gamma|v_2\rangle, \\ |v_1\rangle &= \sqrt{1 - \gamma^2}|\psi_1\rangle + \gamma|v_2\rangle.\end{aligned}$$

It follows from $\langle u_0|u_1 \rangle = 0$ that $\langle v_0|v_1 \rangle = 0 = (1 - \gamma^2)\langle \psi_0|\psi_1 \rangle + \gamma^2$. We denote

$$c^2 \equiv \langle \psi_0|\psi_1 \rangle = -\frac{\gamma^2}{1 - \gamma^2}.$$

Similarly to the RRA scheme, we couple the original system \mathcal{H}_2 to the auxiliary system \mathcal{H}_A by a joint unitary transformation U_3 such that $U_3|v_2 \rangle = |2 \rangle|0_a \rangle$ and

$$\begin{aligned} U_3|\psi_0 \rangle &= \sqrt{1 - |c|^2}|+\rangle|0_a \rangle + \bar{c}|1 \rangle|1_a \rangle, \\ U_3|\psi_1 \rangle &= \sqrt{1 - |c|^2}|-\rangle|0_a \rangle + c|1 \rangle|1_a \rangle. \end{aligned}$$

Thus, we have

$$\begin{aligned} U_3|u_0 \rangle|k_a \rangle &= \sqrt{1 - 2\gamma^2}|+\rangle|0_a \rangle + \sqrt{-\gamma^2}|1 \rangle|1_a \rangle + \gamma|2 \rangle|0_a \rangle, \\ U_3|u_1 \rangle|k_a \rangle &= \sqrt{1 - 2\gamma^2}|-\rangle|0_a \rangle + \sqrt{-\gamma^2}|1 \rangle|1_a \rangle + \gamma|2 \rangle|0_a \rangle \\ U_3|u_2 \rangle|k_a \rangle &= |2 \rangle|0_a \rangle. \end{aligned} \tag{2.1}$$

After the joint transformation, the quantum state we consider in discrimination is given by

$$\rho_\gamma = \sum_{i=0}^2 p_i U_3(|u_i \rangle \langle u_i| \otimes |k_a \rangle \langle k_a|) U_3^\dagger. \tag{2.2}$$

By performing a von Neumann measurement on the auxiliary system by basis, $\{|0_a \rangle \langle 0_a|, |1_a \rangle \langle 1_a|\}$, the state in (2.2) will collapse to either $|0_a \rangle \langle 0_a|$ or $|1_a \rangle \langle 1_a|$. If the system collapses to $|0_a \rangle \langle 0_a|$, we will discriminate the original state since those two states $|u_0 \rangle, |u_1 \rangle$ can be decided completely by the states $|\pm \rangle$ and the state $|u_2 \rangle$ be decided uncertainly by the state $|2 \rangle$ in (2.1). If the qubit collapses to $|1_a \rangle \langle 1_a|$, then we can only decide that the state is not $|u_2 \rangle$. when the qubit collapses to $|1_a \rangle \langle 1_a|$. Thus, we can design a mixed form of ambiguous and unambiguous discriminations as follows:

$$\begin{aligned} \pi_0 &= |+\rangle \langle +| \otimes |0_a \rangle \langle 0_a|, & \pi_1 &= |-\rangle \langle -| \otimes |0_a \rangle \langle 0_a|, & \pi_2 &= |2 \rangle \langle 2| \otimes |0_a \rangle \langle 0_a| \\ \text{and} & & \pi_{fail} &= \mathbb{1}_{\mathcal{H}_2} \otimes |1_a \rangle \langle 1_a|, \end{aligned} \tag{2.3}$$

and the success probability of $\{|u_i \rangle\}_{i=0,1,2}$ can be discriminated is

$$P_{suc} = (1 - 2\gamma^2)(p_0 + p_1) + p_2 = 1 - 2\gamma^2(1 - p_2).$$

Moreover, we have

Theorem 2.1. *Let $\mathcal{H}_2 = \mathbb{C}^3$ and prepare three states $\{|u_i \rangle\}_{i=0,1,2}$ in \mathcal{H}_2 with a priori probability distribution $p = (p_i)$, $\langle u_2|u_0 \rangle = \langle u_2|u_1 \rangle = \gamma$, $\langle u_0|u_1 \rangle = 0$, where γ is a real number and $\gamma \neq 0$. If $p_2 \geq \frac{1}{3}$, then*

$$P_{suc} > P_{suc,max}^{una}.$$

Proof. Following (1.3), we consider a unambiguous discrimination for those three states $\{|u_i\rangle : i = 0, 1, 2\}$ with a priori probability distribution $p = \{p_i\}_i$ by coupling $\mathcal{H}_2 = \mathbb{C}^3$ to \mathcal{H}_A by the joint unitary transformation U_2 as following:

$$U_2|u_i\rangle|k_a\rangle = \sqrt{1 - |\alpha_i|^2}|i\rangle|0_a\rangle + \alpha_i|\phi_i\rangle|1_a\rangle, \quad (2.4)$$

where $\{|\phi_i\rangle, i = 0, 1, 2\} \subseteq \mathcal{H}_2$, and satisfy that $\overline{\alpha_2}\alpha_0\langle\phi_2|\phi_0\rangle = \overline{\alpha_2}\alpha_1\langle\phi_2|\phi_1\rangle = \gamma$ and $\langle\phi_0|\phi_1\rangle = 0$. Now, we decompose $\alpha_2|\phi_2\rangle = \alpha'_0|\phi_0\rangle + \alpha'_1|\phi_1\rangle + \beta|\varphi\rangle$, where $\overline{\alpha'_0}\alpha_0 = \overline{\alpha'_1}\alpha_1 = \gamma$ and $\langle\phi_1|\varphi\rangle = \langle\phi_2|\varphi\rangle = 0$. Then, the success probability of unambiguous discrimination is given by

$$P_{suc}^{una} = 1 - p_0|\alpha_0|^2 - p_1|\alpha_1|^2 - p_2(|\alpha'_0|^2 + |\alpha'_1|^2 + |\beta|^2).$$

Note that the success probability of discrimination is the largest when $\beta = 0$, thus, we find the optimal measurement. Therefore, we can rewrite (2.4) as

$$\begin{aligned} U_2|u_0\rangle|k_a\rangle &= \sqrt{1 - |\alpha_0|^2}|0\rangle|0_a\rangle + \alpha_0|\phi_0\rangle|1_a\rangle, \\ U_2|u_1\rangle|k_a\rangle &= \sqrt{1 - |\alpha_1|^2}|1\rangle|0_a\rangle + \alpha_1|\phi_1\rangle|1_a\rangle, \\ U_2|u_2\rangle|k_a\rangle &= \sqrt{1 - |\alpha'_0|^2 - |\alpha'_1|^2}|2\rangle|0_a\rangle + \alpha'_0|\phi_0\rangle|1_a\rangle + \alpha'_1|\phi_1\rangle|1_a\rangle \end{aligned}$$

where $\overline{\alpha'_0}\alpha_0 = \overline{\alpha'_1}\alpha_1 = \gamma$. The success probability of unambiguous discrimination is given by

$$P_{suc}^{una} = 1 - p_0|\alpha_0|^2 - p_1|\alpha_1|^2 - p_2(|\alpha'_0|^2 + |\alpha'_1|^2). \quad (2.5)$$

Then, by $\overline{\alpha'_0}\alpha_0 = \overline{\alpha'_1}\alpha_1 = \gamma$ and $\max\{|\alpha_0|, |\alpha_1|, |\alpha'_0|, |\alpha'_1|\} \leq 1$, we have that

$$P_{suc}^{una} < 1 - p_0\gamma^2 - p_1\gamma^2 - 2p_2\gamma^2 = 1 - \gamma^2(1 + p_2).$$

This showed that $P_{suc}^{una} < P_{suc}$ when $p_2 \geq \frac{1}{3}$. The success probability (2.5) is applied in any unambiguous discrimination for the states $\{|u_i\rangle : i = 0, 1, 2\}$, thus we have $P_{suc,max}^{una} < P_{suc}$ when $p_2 \geq \frac{1}{3}$. \square

Remark 2.2. When $p_0 = p_1$, ρ_γ is the state of separable form as follows

$$\rho_\gamma = \{1 - \gamma^2(1 - p_2)\}\rho_1^{\mathcal{H}_2} \otimes |0_a\rangle\langle 0_a| + \gamma^2(1 - p_2)|1\rangle\langle 1| \otimes \rho_2^{\mathcal{H}_A}, \quad (2.6)$$

where $\rho_1^{\mathcal{H}_2}$ and $\rho_2^{\mathcal{H}_A}$ are the density matrices of the principal system and the auxiliary system respectively,

$$\begin{aligned} \rho_1^{\mathcal{H}_2} &= \frac{1}{1 - (1 - p_2)\gamma^2} \left\{ \frac{1}{2}(1 - p_2)(1 - 2\gamma^2)(|+\rangle\langle +| + |-\rangle\langle -|) + ((1 - p_2)\gamma^2 + p_2)|2\rangle\langle 2| \right. \\ &\quad \left. + \frac{\sqrt{2}}{2}(1 - p_2)\gamma\sqrt{(1 - 2\gamma^2)}(|0\rangle\langle 2| + |2\rangle\langle 0|) \right\}, \\ \rho_2^{\mathcal{H}_A} &= \frac{1}{(1 - p_2)\gamma^2} \left\{ (1 - p_2)\gamma^2|1_a\rangle\langle 1_a| + \frac{\sqrt{2}}{2}(1 - p_2)\sqrt{-\gamma^2(1 - 2\gamma^2)}|0_a\rangle\langle 1_a| \right. \\ &\quad \left. + \frac{\sqrt{2}}{2}(1 - p_2)\sqrt{-\gamma^2(1 - 2\gamma^2)}|1_a\rangle\langle 0_a| \right\}. \end{aligned}$$

Thus, the discrimination of three states can be performed with the absence of entanglement. And, from (2.6) and the necessary and sufficient condition of zero discord in Ref. [7], we have zero left quantum discord because that $[\rho_1^{\mathcal{H}_2}, |1\rangle\langle 1|] = 0$. But, if $|\gamma| \neq \frac{1}{\sqrt{2}}$, the right discord is non-zero.

3 Generalization of the mixed form discrimination

Next, we consider a general case, that is, let $\langle u_2|u_0\rangle = \langle u_2|u_1\rangle = \gamma$, $\langle u_0|u_1\rangle = \alpha$, where γ, α be real numbers, and $\gamma \neq 0, 1; \alpha \neq 0, 1$. Let us define

$$|v_i\rangle \equiv |u_i\rangle|k_a\rangle.$$

Taking two states $|\psi_3\rangle, |\psi_4\rangle$ such that $\langle v_2|\psi_3\rangle = \langle v_2|\psi_4\rangle = 0$, and

$$\begin{aligned} |v_0\rangle &= \sqrt{1-\gamma^2}|\psi_3\rangle + \gamma|v_2\rangle, \\ |v_1\rangle &= \sqrt{1-\gamma^2}|\psi_4\rangle + \gamma|v_2\rangle. \end{aligned}$$

Note that $\langle v_0|v_1\rangle = \alpha = (1-\gamma^2)\langle\psi_3|\psi_4\rangle + \gamma^2$, we denote

$$c^2 = \langle\psi_3|\psi_4\rangle = \frac{\alpha - \gamma^2}{1 - \gamma^2}.$$

Now, we couple $\mathcal{H}_2 = \mathbb{C}^3$ to \mathcal{H}_A by a joint unitary transformation U_4 such that $U_4|v_2\rangle = |2\rangle|0_a\rangle$ and

$$\begin{aligned} U_4|\psi_3\rangle &= \sqrt{1-|c|^2}|+\rangle|0_a\rangle + \bar{c}|1\rangle|1_a\rangle, \\ U_4|\psi_4\rangle &= \sqrt{1-|c|^2}|-\rangle|0_a\rangle + c|1\rangle|1_a\rangle. \end{aligned}$$

Thus, we have

$$\begin{aligned} U_4|u_0\rangle|k_a\rangle &= \sqrt{1-\gamma^2-|\alpha-\gamma^2|}|+\rangle|0_a\rangle + \sqrt{\alpha-\gamma^2}|1\rangle|1_a\rangle + \gamma|2\rangle|0_a\rangle, \\ U_4|u_1\rangle|k_a\rangle &= \sqrt{1-\gamma^2-|\alpha-\gamma^2|}|-\rangle|0_a\rangle + \sqrt{\alpha-\gamma^2}|1\rangle|1_a\rangle + \gamma|2\rangle|0_a\rangle, \\ U_4|u_2\rangle|k_a\rangle &= |2\rangle|0_a\rangle. \end{aligned}$$

After the joint transformation, the quantum state we consider in discrimination is given by

$$\rho_{\gamma,\alpha} = \sum_{i=0}^2 p_i U_4(|u_i\rangle\langle u_i| \otimes |k_a\rangle\langle k_a|) U_4^\dagger. \quad (3.1)$$

Then, when $\alpha < \gamma^2$, by performing the von Neumann measurement such as (2.3), the success probability of $\{|u_i\rangle\}_{i=0,1,2}$ can be discriminated is

$$P_{suc,\alpha} = 1 - (2\gamma^2 - \alpha)(1 - p_2),$$

when $\alpha \geq \gamma^2$, the success probability is

$$P_{suc,\alpha} = 1 - \alpha(1 - p_2). \quad (3.2)$$

Remark 3.1. When $\alpha < \gamma^2$ and $p_0 = p_1$, the quantum state (3.1) is the state of separable form as follows

$$\rho_{\gamma,\alpha} = \{1 - (1 - p_2)(\gamma^2 - \alpha)\}\rho_3^{\mathcal{H}_2} \otimes |0_a\rangle\langle 0_a| + (1 - p_2)(\gamma^2 - \alpha)|1\rangle\langle 1| \otimes \rho_4^{\mathcal{H}_A}, \quad (3.3)$$

where $\rho_1^{\mathcal{H}_2}$ and $\rho_2^{\mathcal{H}_A}$ are the density matrices of the principal system and the auxiliary system respectively,

$$\begin{aligned} \rho_3^{\mathcal{H}_2} &= \frac{1}{1 - (1 - p_2)(\gamma^2 - \alpha)} \left\{ \frac{1}{2}(1 - p_2)(1 + \alpha - 2\gamma^2)(|+\rangle\langle +| + |-\rangle\langle -|) \right. \\ &\quad \left. + ((1 - p_2)\gamma^2 + p_2)|2\rangle\langle 2| + \frac{\sqrt{2}}{2}(1 - p_2)\gamma\sqrt{(1 + \alpha - 2\gamma^2)}(|0\rangle\langle 2| + |2\rangle\langle 0|) \right\}, \\ \rho_4^{\mathcal{H}_A} &= \frac{1}{\gamma^2 - \alpha} \left\{ (\gamma^2 - \alpha)|1_a\rangle\langle 1_a| + \frac{\sqrt{2}}{2}\sqrt{(\alpha - \gamma^2)(1 + \alpha - 2\gamma^2)}|0_a\rangle\langle 1_a| \right. \\ &\quad \left. + \frac{\sqrt{2}}{2}\sqrt{(\alpha - \gamma^2)(1 + \alpha - 2\gamma^2)}|1_a\rangle\langle 0_a| \right\}. \end{aligned}$$

Then, as Remark 2.2, the discrimination of three states can be performed with the absence of entanglement. And, from (3.3) and the necessary and sufficient condition of zero discord in [7], we have zero left quantum discord because that $[\rho_3^{\mathcal{H}_2}, |1\rangle\langle 1|] = 0$. But, the right discord is non-zero.

Theorem 3.2. Let $\langle u_i | u_{j \neq i} \rangle = \gamma$ for $i, j = 0, 1, 2$, then

$$P_{suc,\gamma} \geq P_{suc,max}^{una}.$$

Proof. Without lose of generality, we can assume $p_2 = \max\{p_i\}_{i=0,1,2}$. By (3.2), we have $P_{suc,\gamma} = 1 - \gamma(1 - p_2) = 1 - \gamma(p_0 + p_1)$.

If the conditions (1) and (2) are satisfied in Example 1.2, then $P_{suc,\gamma} \geq P_{suc,max}^{una}$ is clear. If the condition (3) is satisfied in Example 1.2, note that $p_0 \geq p_0\gamma$ and $2\sqrt{p_1p_2} \geq p_1$, thus $P_{suc,\gamma} \geq P_{suc,max}^{una}$. If the condition (4) is satisfied in Example 1.2, note that the following inequalities:

$$p_0 \leq \sqrt{p_1p_2}, \quad p_1 \leq \sqrt{p_1p_2} \quad \text{and} \quad \sqrt{p_0p_1} \leq \sqrt{p_0p_2}$$

where $p_0 \leq p_1 \leq p_2$, we have that

$$\begin{aligned} P_{suc,\gamma} &= 1 - (p_0 + p_1)\gamma \geq 1 - 2\sqrt{p_1 p_2}\gamma \\ &\geq 1 - 2(\sqrt{p_1 p_2} + \sqrt{p_0 p_2} - \sqrt{p_0 p_1})\gamma = P_{suc,max}^{una}. \end{aligned}$$

□

Remark 3.3. When $\alpha = \gamma^2$, it is possible to perform the above discrimination even without the auxiliary qubit system, because that the discrimination can be performed with the absence of both entanglement and quantum discord. This is also applied to following case:

Let $\langle u_i | u_{j \neq i} \rangle = \gamma_{ij}$ satisfy that $\overline{\gamma_{12}\gamma_{20}} = \gamma_{01}$. Take two quantum states $|\psi_0\rangle, |\psi_1\rangle$ such that $\langle u_2 | \psi_0 \rangle = \langle u_2 | \psi_1 \rangle = 0$ and

$$\begin{aligned} |u_0\rangle &= \sqrt{1 - |\gamma_{20}|^2} |\psi_0\rangle + \gamma_{20} |u_2\rangle, \\ |u_1\rangle &= \sqrt{1 - |\gamma_{12}|^2} |\psi_1\rangle + \overline{\gamma_{12}} |u_2\rangle. \end{aligned}$$

Thus, we have $\langle \psi_0 | \psi_1 \rangle = 0$ since $\overline{\gamma_{12}\gamma_{20}} = \gamma_{01}$. Let us perform the measurement $\Pi_4 = \{\pi_i\}_i$ defined by

$$\pi_0 = |\psi_0\rangle\langle\psi_0|, \quad \pi_1 = |\psi_1\rangle\langle\psi_1| \quad \text{and} \quad \pi_2 = |u_2\rangle\langle u_2|$$

on the state $\rho = \sum_{i=0}^3 p_i |u_i\rangle\langle u_i|$. Then, those two states $|u_0\rangle, |u_1\rangle$ can be decided completely when outcome is $i = 0, 1$, although the state $|u_2\rangle$ cannot be decided completely, but, we can decide it in following probability:

$$\frac{p_2 \text{Tr}(\pi_2 |u_2\rangle\langle u_2|)}{p_0 \text{Tr}(\pi_2 |u_0\rangle\langle u_0|) + p_1 \text{Tr}(\pi_2 |u_1\rangle\langle u_1|) + p_2 \text{Tr}(\pi_2 |u_2\rangle\langle u_2|)} = \frac{p_2}{p_0 |\gamma_{20}|^2 + p_1 |\gamma_{12}|^2 + p_2}.$$

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